

Optimum allocation in multivariate stratified random sampling: Stochastic matrix optimisation

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Abstract

The allocation problem for multivariate stratified random sampling as a problem of stochastic matrix integer mathematical programming is considered. With these aims the asymptotic normality of sample covariance matrices for each strata is established. Some alternative approaches are suggested for its solution. An example is solved by applying the proposed techniques.

Key Words: Multivariate stratified random sampling, modified E -model, stochastic programming, optimum allocation, integer programming, E -model, V -model, P -model.

Mathematics Subject Classification: 62D05, 90C15, 90C29, 90C10

1 INTRODUCTION

Not long ago, multivariate analysis was mainly based on linear methods illustrated on small to medium-sized data sets. However, many novel developments, have permitted the introduction of several innovative statistical and mathematical tools for high-dimensional data analysis. Developments as generalised multivariate analysis, latent variable analysis, DNA microarray data, pattern recognition, multivariate nonlinear analysis, data mining, manifold learning, shape theory etc., have given a new and modern image to Multivariate Analysis.

One of the topics of statistical theory that is most commonly used in many fields of scientific research is the theory of probabilistic sampling. From a multivariate point of view, diverse authors have studied the problem of optimum allocation in multivariate stratified random sampling. Arthanari and Dodge (1981) and Sukhatme *et al.* (1984), among many others, proposed the problem of optimum allocation in multivariate stratified random sampling as a deterministic multiobjective mathematical programming problem, by considering as objective function a cost function subject to restrictions on certain functions of variances or viceversa, i.e., considering the functions of variances as objective and subject to restrictions on costs. Noting that, for the case when the function of costs is taken as the objective function, the problem of optimum allocation in multivariate stratified random sampling is reduced to a classical uniobjective mathematical programming problem.

Furthermore, Díaz-García and Ulloa (2008) propose the optimum allocation in multivariate stratified random sampling as a deterministic nonlinear problem of matrix integer mathematical programming constrained by a cost function or by a given sample size. Also, Prékopa (1978) and Díaz-García and Garay (2007) observe that the values of the population variances are in fact random variables and formulate the corresponding problem of optimum allocation in multivariate stratified random sampling as a stochastic mathematical programming problem.

In this paper, the optimum allocation in multivariate stratified random sampling is posed as a stochastic matrix integer mathematical programming problem constrained by a cost function or by a given sample size. Section 2 provides notation and definitions on multivariate stratified random sampling. Section 3 studies in detail the asymptotic normality of the sample mean vectors and covariance matrices. The optimum allocation in multivariate stratified random sampling via stochastic matrix integer mathematical programming is given in Section 4. Also, several particular solutions are derived for solving the proposed stochastic mathematical programming problems. Finally, an example of the literature is given in Section 5.

2 PRELIMINARY RESULTS ON MULTIVARIATE STRATIFIED RANDOM SAMPLING

Consider a population of size N , divided into H sub-populations (strata). We wish to find a representative sample of size n and an optimum allocation in the strata meeting the following requirements: i) to minimise the variance of the estimated mean subject to a budgetary constraint; or ii) to minimise the cost subject to a constraint on the variances; this is the classical problem in optimum allocation in univariate stratified sampling, see Cochran (1977), Sukhatme *et al.* (1984) and Thompson (1997). However, if more than one characteristic (variable) is being considered then the problem is known as optimum allocation in multivariate stratified sampling. For a formal expression of the problem of optimum allocation in stratified sampling, consider the following notation.

The subindex $h = 1, 2, \dots, H$ denotes the stratum, $i = 1, 2, \dots, N_h$ or n_h the unit within stratum h and $j = 1, 2, \dots, G$ denotes the characteristic (variable). Moreover:

N_h	Total number of units within stratum h .
n_h	Number of units from the sample in stratum h .
$\mathbf{Y}_h = (\mathbf{Y}_h^1, \dots, \mathbf{Y}_h^G)$	$N_h \times G$ population matrix in stratum h ; \mathbf{Y}_{hi} is the
$= (\mathbf{Y}_{h1}, \dots, \mathbf{Y}_{hN_h})'$	G -dimensional value of the i -th unit in stratum h .
$\mathbf{y}_h = (\mathbf{y}_h^1, \dots, \mathbf{y}_h^G)$	$n_h \times G$ sample matrix in stratum h ; \mathbf{y}_{hi} is the G -dimensional
$= (\mathbf{y}_{h1}, \dots, \mathbf{y}_{hn_h})'$	G -dimensional value of the i -th unit of the sample in stratum h .
y_{hi}^j	Value obtained for the i -th unit in stratum h of the j -th characteristic
$\mathbf{n} = (n_1, \dots, n_H)'$	Vector of the number of units in the sample
$W_h = \frac{N_h}{N}$	Relative size of stratum h
$\bar{Y}_h^j = \frac{1}{N_h} \sum_{i=1}^{N_h} y_{hi}^j$	Population mean in stratum h of the j -th characteristic.
$\bar{\mathbf{Y}}_h = (\bar{Y}_h^1, \dots, \bar{Y}_h^G)'$	Population mean vector in stratum h .
$\bar{y}_h^j = \frac{1}{n_h} \sum_{i=1}^{n_h} y_{hi}^j$	Sample mean in stratum h of the j -th characteristic.
$\bar{\mathbf{y}}_h = (\bar{y}_h^1, \dots, \bar{y}_h^G)'$	Sample mean vector in stratum h .

$$\bar{y}_{ST}^j = \sum_{h=1}^H W_h \bar{y}_h^j$$

$$\bar{\mathbf{y}}_{ST} = (\bar{y}_{ST}^1, \dots, \bar{y}_{ST}^G)'$$

\mathbf{S}_h

Estimator of the population mean in multivariate

stratified sampling for the j -th characteristic.

Estimator of the population mean vector in multivariate stratified sampling.

Covariance matrix in stratum h

$$\mathbf{S}_h = \frac{1}{N_h} \sum_{i=1}^{N_h} (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)'$$

where $S_{h_{jk}}$ is the covariance in stratum h of the j -th and k -th characteristics; furthermore

$$S_{h_{jk}} = \frac{1}{N_h} \sum_{i=1}^{N_h} (y_{hi}^j - \bar{y}_h^j)(y_{hi}^k - \bar{y}_h^k), \text{ and}$$

$$S_{h_{jj}} \equiv S_{h_j}^2 = \frac{1}{N_h} \sum_{i=1}^{N_h} (y_{hi}^j - \bar{y}_h^j)^2.$$

\mathbf{s}_h

Estimator of the covariance matrix in stratum h ;

$$\mathbf{s}_h = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)'$$

where $s_{h_{jk}}$ is the sample covariance in stratum h of the j -th and k -th characteristics; furthermore

$$s_{h_{jk}} = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (y_{hi}^j - \bar{y}_h^j)(y_{hi}^k - \bar{y}_h^k), \text{ and}$$

$$s_{h_{jj}} \equiv s_{h_j}^2 = \frac{1}{n_h - 1} \sum_{i=1}^{n_h} (y_{hi}^j - \bar{y}_h^j)^2.$$

$\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$

$\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$

Covariance matrix of $\bar{\mathbf{y}}_{ST}$.

Estimator of the covariance matrix of $\bar{\mathbf{y}}_{ST}$,

it is denoted as $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \equiv \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$, and defined as

$$= \begin{pmatrix} \widehat{\text{Var}}(\bar{y}_{ST}^1) & \widehat{\text{Cov}}(\bar{y}_{ST}^1, \bar{y}_{ST}^2) & \dots & \widehat{\text{Cov}}(\bar{y}_{ST}^1, \bar{y}_{ST}^G) \\ \widehat{\text{Cov}}(\bar{y}_{ST}^2, \bar{y}_{ST}^1) & \widehat{\text{Var}}(\bar{y}_{ST}^2) & \dots & \widehat{\text{Cov}}(\bar{y}_{ST}^2, \bar{y}_{ST}^G) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{\text{Cov}}(\bar{y}_{ST}^G, \bar{y}_{ST}^1) & \widehat{\text{Cov}}(\bar{y}_{ST}^G, \bar{y}_{ST}^2) & \dots & \widehat{\text{Var}}(\bar{y}_{ST}^G) \end{pmatrix}$$

$$= \sum_{h=1}^H \frac{W_h^2 \mathbf{s}_h}{n_h} - \sum_{h=1}^H \frac{W_h \mathbf{s}_h}{N}$$

$\widehat{\text{Cov}}(\bar{y}_{ST}^j, \bar{y}_{ST}^k)$

Estimated covariance of \bar{y}_{ST}^j and \bar{y}_{ST}^k where

$\widehat{\text{Cov}}(\bar{y}_{ST}^j, \bar{y}_{ST}^k) \equiv \widehat{\text{Cov}}(\bar{y}_{ST}^j, \bar{y}_{ST}^k)$, with

$$\widehat{\text{Cov}}(\bar{y}_{ST}^j, \bar{y}_{ST}^k) = \sum_{h=1}^H \frac{W_h^2 s_{h_{jk}}}{n_h} - \sum_{h=1}^H \frac{W_h s_{h_{jk}}}{N}, \text{ and}$$

$$\widehat{\text{Cov}}(\bar{y}_{ST}^j, \bar{y}_{ST}^j) \equiv \widehat{\text{Var}}(\bar{y}_{ST}^j) = \sum_{h=1}^H \frac{W_h^2 s_{h_j}^2}{n_h} - \sum_{h=1}^H \frac{W_h s_{h_j}^2}{N}.$$

c_h

Cost per G -dimensional sampling unit in stratum h and let

$\mathbf{c} = (c_1, \dots, c_G)'$.

Where if $\mathbf{a} \in \mathbb{R}^G$, \mathbf{a}' denotes the transpose of \mathbf{a} .

3 LIMITING DISTRIBUTION OF SAMPLE MEANS AND COVARIANCE MATRICES

In this section the asymptotic distribution of the estimator of the covariance matrix \mathbf{s}_h and mean $\bar{\mathbf{y}}_h$ is considered. With this aim in mind, the multivariate version of Hájek's theorem

is proposed in the context of sampling theory in terms of the extension stated in Hájek (1961). First, consider the following notation and definitions.

A detailed discussion of operator “vec”, “vech”, Moore-Penrose inverse, Kronecker product, commutation matrix and duplication matrix may be found in Magnus and Neudecker (1988), among many others. For convenience, some notations shall be introduced, although in general it adheres to standard notations.

For all matrix \mathbf{A} , there exists a unique matrix \mathbf{A}^+ which is termed the *Moore-Penrose inverse* of \mathbf{A} .

Let \mathbf{A} be an $m \times n$ matrix and \mathbf{B} a $p \times q$ matrix. The $mp \times nq$ matrix defined by

$$\begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix}$$

is termed the *Kronecker product* (also termed tensor product or direct product) of \mathbf{A} and \mathbf{B} and written $\mathbf{A} \otimes \mathbf{B}$. Let \mathbf{C} be an $m \times n$ matrix and \mathbf{C}_j its j -th column, then $\text{vec } \mathbf{C}$ is the $mn \times 1$ vector

$$\text{vec } \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \vdots \\ \mathbf{C}_n \end{bmatrix}.$$

The vector $\text{vec } \mathbf{C}$ and $\text{vec } \mathbf{C}'$ clearly contain the same mn components, but in different order. Therefore there exist a unique $mn \times mn$ permutation matrix which transform $\text{vec } \mathbf{C}$ into $\text{vec } \mathbf{C}'$. This matrix is termed the *commutation matrix* and is denoted \mathbf{K}_{mn} . (If $m = n$, is often write \mathbf{K}_n instead of \mathbf{K}_{mn} .) Hence

$$\mathbf{K}_{mn} \text{vec } \mathbf{C} = \text{vec } \mathbf{C}'.$$

Similarly, let \mathbf{B} be a square $n \times n$ matrix. Then $\text{vech } \mathbf{B}$ (also denoted as $v(\mathbf{B})$) shall denote the $n(n+1)/2 \times 1$ vector that is obtained from $\text{vec } \mathbf{B}$ by eliminating all supradiagonal elements of \mathbf{B} . If $\mathbf{B} = \mathbf{B}'$, $\text{vech } \mathbf{B}$ contains only the distinct elements of \mathbf{B} , then there is a unique $n^2 \times n(n+1)/2$ matrix termed *duplication matrix*, which is denoted by \mathbf{D}_n , such that $\mathbf{D}_n \text{vech } \mathbf{B} = \text{vec } \mathbf{B}$ and $\mathbf{D}_n^+ \text{vec } \mathbf{B} = \text{vech } \mathbf{B}$. Finally, denote $(\text{vech } \mathbf{B})' \equiv \text{vech}' \mathbf{B}$.

In what follows, from Lemma 3.1 through Theorem 3.2, asymptotic results are stated for a single stratum. The notation N_ν and n_ν denote the size of a generic stratum and the size of a simple random sample from that stratum.

Lemma 3.1. *Let Ξ_ν be a $G \times G$ symmetric random matrix defined as*

$$\Xi_\nu = \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)'$$

Suppose that for $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k)'$, any vector of constants, $k = G(G+1)/2$,

$$\boldsymbol{\lambda}' (\mathbf{M}_\nu^4 - \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu) \boldsymbol{\lambda} \geq \epsilon \max_{1 \leq \alpha \leq k} \left[\lambda_\alpha^2 \mathbf{e}_k^{\alpha'} (\mathbf{M}_\nu^4 - \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu) \mathbf{e}_k^\alpha \right], \quad (1)$$

where $\mathbf{e}_k^\alpha = (0, \dots, 0, 1, 0, \dots, 0)'$ is the α -th vector of the canonical base of \mathbb{R}^k , $\epsilon > 0$ and independent of $\nu > 1$ and

$$\mathbf{M}_\nu^4 = \frac{1}{N_\nu} \mathbf{D}_G^+ \left[\sum_{i=1}^{N_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \right] \mathbf{D}_G^{+'},$$

is the fourth central moment. Assume that $n_\nu \rightarrow \infty$, $N_\nu - n_\nu \rightarrow \infty$, $N_\nu \rightarrow \infty$, and that, for all $j = 1, \dots, G$,

$$\left[\lim_{\nu \rightarrow \infty} \left(\frac{n_\nu}{N_\nu} \right) = 0 \right] \Rightarrow \lim_{\nu \rightarrow \infty} \frac{\max_{1 \leq i_1 < \dots < i_{n_\nu} \leq N_\nu} \sum_{\beta=1}^{n_\nu} \left[\left(y_{\nu i_\beta}^j - \bar{Y}_\nu^j \right)^2 - S_{\nu j}^2 \right]^2}{N_\nu \left[m_{\nu j}^4 - \left(S_{\nu j}^2 \right)^2 \right]} = 0, \quad (2)$$

where

$$m_{\nu j}^4 = \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} \left(y_{\nu i}^j - \bar{y}_\nu^j \right)^4.$$

Then, $\text{vech } \Xi_\nu$ is asymptotically normally distributed as

$$\text{vech } \Xi_\nu \xrightarrow{d} \mathcal{N}_k(\mathbf{E}(\text{vech } \Xi_\nu), \text{Cov}(\text{vech } \Xi_\nu)),$$

with

$$\mathbf{E}(\text{vech } \Xi_\nu) = \frac{n_\nu}{n_\nu - 1} \text{vech } \mathbf{S}_\nu, \quad (3)$$

and

$$\text{Cov}(\text{vech } \Xi_\nu) = \frac{n_\nu}{(n_\nu - 1)^2} (\mathbf{M}_\nu^4 - \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu). \quad (4)$$

n_ν is the sample size for a simple random sample from the ν -th population of size N_ν .

Remark 3.1. Let

$$\Xi_\nu = \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)'.$$

Hence,

$$\begin{aligned} \text{vec } \Xi_\nu &= \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} \text{vec}(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \\ &= \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu). \end{aligned}$$

From where

$$\text{vech } \Xi_\nu = \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} \mathbf{D}_G^+ (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu),$$

$k = G(G + 1)/2$.

Taking $m = k$ and $\mathbf{a}_{\nu i} = (a_{\nu i}^1, \dots, a_{\nu i}^k) = \mathbf{D}_G^+ (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)$ in Hájek (1961), it is obtained that:

i) $\text{vech } \Xi_\nu$ can be expressed as

$$\text{vech } \Xi_\nu = \sum_{i=1}^{N_\nu} b_{\nu i} \mathbf{a}_{\nu R_{\nu i}}.$$

with b 's fixed, furthermore $b_{\nu 1} = \dots = b_{\nu n_\nu} = 1/(n_\nu - 1)$, $b_{\nu n_\nu+1} = \dots = b_{\nu N_\nu} = 0$. Then

$$\lim_{\nu \rightarrow \infty} \frac{\max_{1 \leq j \leq N_\nu} (b_{\nu j} - \bar{b}_\nu)^2}{\sum_{i=1}^{N_\nu} (b_{\nu i} - \bar{b}_\nu)^2} = 0, \quad \text{where} \quad \bar{b}_\nu = \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} b_{\nu i}$$

holds if $n_\nu \rightarrow \infty$, $N_\nu - n_\nu \rightarrow \infty$.

ii) $\bar{\mathbf{a}}_\nu = (\bar{a}_\nu^1 \dots \bar{a}_\nu^k)'$ is

$$\begin{aligned} \bar{\mathbf{a}}_\nu &= \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} \mathbf{a}_{\nu i} \\ &= \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} \mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \\ &= \text{vech } \frac{1}{N_\nu} \sum_{i=1}^{N_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \\ &= \text{vech } \mathbf{S}_\nu \end{aligned}$$

iii) From (7.2) in Hájek (1961)

$$\sum_{i=1}^{N_\nu} \left[\sum_{\alpha=1}^k \lambda_\alpha (a_{\nu i}^\alpha - a_\nu^\alpha) \right]^2 \geq \epsilon \max_{1 \leq \alpha \leq k} \left[\lambda_\alpha^2 \sum_{i=1}^{N_\nu} (a_{\nu i}^\alpha - a_\nu^\alpha)^2 \right]. \quad (5)$$

In the context of sampling theory the right side in (5) can be written as

$$\begin{aligned} \sum_{i=1}^{N_\nu} \left[\sum_{\alpha=1}^k \lambda_\alpha (a_{\nu i}^\alpha - a_\nu^\alpha) \right]^2 &= \sum_{i=1}^{N_\nu} \{ \boldsymbol{\lambda}' [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) - \text{vech } \mathbf{S}_\nu] \}^2 \\ &= \sum_{i=1}^{N_\nu} \boldsymbol{\lambda}' [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) - \text{vech } \mathbf{S}_\nu] \\ &\quad \left[(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \mathbf{D}_G^{+'} - \text{vech}' \mathbf{S}_\nu \right] \boldsymbol{\lambda} \\ &= \boldsymbol{\lambda}' \left[\mathbf{D}_G^+ \sum_{i=1}^{N_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \mathbf{D}_G^{+'} \right. \\ &\quad \left. - \text{vech } \mathbf{S}_\nu \sum_{i=1}^{N_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \mathbf{D}_G^{+'} \right. \\ &\quad \left. - \mathbf{D}_G^+ \sum_{i=1}^{N_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \text{vech}' \mathbf{S}_\nu + N_\nu \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu \right] \boldsymbol{\lambda} \\ &= N_\nu \boldsymbol{\lambda}' (\mathbf{M}_\nu^4 - \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu) \boldsymbol{\lambda}, \end{aligned} \quad (6)$$

where \mathbf{M}_ν^4 is

$$= \frac{1}{N_\nu} \mathbf{D}_G^+ \left[\sum_{i=1}^{N_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \right] \mathbf{D}_G^{+'}, \quad (7)$$

Similarly the right side of (5) is

$$\begin{aligned} \lambda_\alpha^2 \sum_{i=1}^{N_\nu} (a_{\nu i}^\alpha - a_\nu^\alpha)^2 &= \sum_{i=1}^{N_\nu} \left\{ \boldsymbol{\lambda}' \mathbf{e}_k^\alpha \mathbf{e}_k^{\alpha'} [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) - \text{vech } \mathbf{S}_\nu] \right\}^2 \\ &= \lambda_\alpha^2 \sum_{i=1}^{N_\nu} \left\{ \mathbf{e}_k^{\alpha'} [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) - \text{vech } \mathbf{S}_\nu] \right\}^2. \end{aligned}$$

Then, proceeding as in 3.,

$$\lambda_\alpha^2 \sum_{i=1}^{N_\nu} (a_{\nu i}^\alpha - a_\nu^\alpha)^2 = N_\nu \lambda_\alpha^2 \mathbf{e}_k^{\alpha'} (\mathbf{M}_\nu^4 - \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu) \mathbf{e}_k^\alpha. \quad (8)$$

Therefore, from (6) and (8), (1) is established.

iv) The expression for (2) is found analogously as the procedure described in item 3.

v) Finally,

$$\begin{aligned} \mathbb{E}(\text{vech } \boldsymbol{\Xi}) &= \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} \mathbb{E} \mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \\ &= \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} \text{vech } \mathbb{E}(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \\ &= \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} \text{vech } \mathbf{S}_\nu \\ &= \frac{n_\nu}{n_\nu - 1} \text{vech } \mathbf{S}_\nu \end{aligned}$$

Similarly, by independence

$$\begin{aligned} \text{Cov}(\text{vech } \boldsymbol{\Xi}) &= \frac{1}{(n_\nu - 1)^2} \sum_{i=1}^{n_\nu} \text{Cov} [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)] \\ &= \frac{1}{(n_\nu - 1)^2} \sum_{i=1}^{n_\nu} \left\{ \mathbb{E} [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \mathbf{D}_G^{+'}] \right. \\ &\quad \left. - \mathbb{E} [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)] \mathbb{E} [(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \mathbf{D}_G^{+'}] \right\} \\ &= \frac{1}{(n_\nu - 1)^2} \sum_{i=1}^{n_\nu} (\mathbf{M}_\nu^4 - \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu) \\ &= \frac{n_\nu}{(n_\nu - 1)^2} (\mathbf{M}_\nu^4 - \text{vech } \mathbf{S}_\nu \text{vech}' \mathbf{S}_\nu), \end{aligned}$$

the last expression is obtained observing that

$$\mathbb{E} [\mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu) \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)] = \text{vech } \mathbb{E} [(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)'] = \text{vech } \mathbf{S}_\nu$$

and that

$$\mathbb{E} \left\{ \mathbf{D}_G^+(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \mathbf{D}_G^{+'} \right\} = \mathbf{M}_\nu^4$$

where \mathbf{M}_ν^4 is defined in (7). \square

Theorem 3.1. *Under assumptions in Lemma 3.1, the sequence of sample covariance matrices \mathbf{s}_ν are such that $\text{vech } \mathbf{s}_\nu$ has an asymptotic normal distribution with asymptotic mean and covariance matrix given by (3) and (4), respectively.*

Proof. This follows immediately from Lemma 3.1, only observe that

$$\begin{aligned} \mathbf{s}_\nu &= \frac{1}{n_\nu - 1} \sum_{i=1}^{n_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{y}}_\nu)' \\ &= \mathbf{\Xi} - \frac{n_\nu}{n_\nu - 1} (\bar{\mathbf{y}}_\nu - \bar{\mathbf{Y}}_\nu)(\bar{\mathbf{y}}_\nu - \bar{\mathbf{Y}}_\nu)', \end{aligned}$$

where

$$\frac{n_\nu}{n_\nu - 1} \rightarrow 1 \quad \text{and} \quad (\bar{\mathbf{y}}_\nu - \bar{\mathbf{Y}}_\nu)(\bar{\mathbf{y}}_\nu - \bar{\mathbf{Y}}_\nu)' \rightarrow 0 \quad \text{in probability.} \quad \square$$

\square

Remark 3.2. Observe that it is possible to find the asymptotic distribution of $\text{vec } \mathbf{s}_\nu$, but this asymptotic normal distribution is singular, because $\text{Cov}(\text{vec } \mathbf{s}_\nu)$ is singular. This is due to the fact $\text{Cov}(\text{vec } \mathbf{s}_\nu)$ is the $G^2 \times G^2$ covariance matrix in the asymptotic distribution of $\text{vec } \mathbf{s}_\nu$ and, because \mathbf{s}_ν is symmetric, then $\text{vec } \mathbf{s}_\nu$ has repeated elements. In this case, $\text{vec } \mathbf{s}_\nu$ is asymptotically normally distributed as (see Muirhead (1982))

$$\text{vec } \mathbf{s}_\nu \xrightarrow{d} \mathcal{N}_{G^2}(\mathbb{E}(\text{vec } \mathbf{\Xi}_\nu), \text{Cov}(\text{vec } \mathbf{\Xi}_\nu)),$$

where

$$\mathbb{E}(\text{vec } \mathbf{\Xi}_\nu) = \frac{n_\nu}{n_\nu - 1} \text{vec } \mathbf{S}_\nu,$$

$$\text{Cov}(\text{vec } \mathbf{\Xi}_\nu) = \frac{n_\nu}{(n_\nu - 1)^2} (\mathfrak{M}_\nu^4 - \text{vec } \mathbf{S}_\nu \text{vec}' \mathbf{S}_\nu),$$

and

$$\mathfrak{M}_\nu^4 = \frac{1}{N_\nu} \left[\sum_{i=1}^{N_\nu} (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \otimes (\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)(\mathbf{y}_{\nu i} - \bar{\mathbf{Y}}_\nu)' \right]. \quad \square$$

Proceeding in analogous way as in Lemma 3.1 and Remark 3.1, it is obtained:

Theorem 3.2. *Suppose that for $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_G)'$, any vector of constants,*

$$\boldsymbol{\lambda}' \mathbf{S}_\nu \boldsymbol{\lambda} \geq \epsilon \max_{1 \leq j \leq G} [\lambda_\alpha^2 S_{\nu\alpha}^2]. \quad (9)$$

Assume that $n_\nu \rightarrow \infty$, $N_\nu - n_\nu \rightarrow \infty$, $N_\nu \rightarrow \infty$, and that

$$\left[\lim_{\nu \rightarrow \infty} \left(\frac{n_\nu}{N_\nu} \right) = 0 \right] \Rightarrow \lim_{\nu \rightarrow \infty} \frac{\max_{1 \leq i_1 < \dots < i_{n_\nu} \leq N_\nu} \sum_{\beta=1}^{n_\nu} (y_{\nu i_\beta}^j - \bar{Y}_\nu^j)^2}{N_\nu S_{\nu j}^2} = 0, \quad (10)$$

Then, $\bar{\mathbf{y}}_\nu$ is asymptotically normally distributed as

$$\bar{\mathbf{y}}_\nu \xrightarrow{d} \mathcal{N}_G(\bar{\mathbf{Y}}_\nu, \mathbf{S}_\nu).$$

n_ν is the sample size for a simple random sample from the ν -th population of size N_ν .

As direct consequence of Theorem 3.1 it is obtained:

Theorem 3.3. Let $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$ be the estimator of the covariance matrix of $\bar{\mathbf{y}}_{ST}$, then

$$\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) = \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \text{vech } \mathbf{s}_h$$

is asymptotically normally distributed; furthermore

$$\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_k \left(\mathbb{E} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right), \quad (11)$$

where

$$\mathbb{E} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) = \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \frac{n_h}{n_h - 1} \text{vech } \mathbf{S}_h, \quad (12)$$

$$\begin{aligned} \text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) &= \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right)^2 \frac{n_h}{(n_h - 1)^2} (\mathbf{M}_h^4 - \text{vech } \mathbf{S}_h \text{vech}' \mathbf{S}_h), \end{aligned} \quad (13)$$

and

$$\mathbf{M}_h^4 = \frac{1}{N_h} \mathbf{D}_G^+ \left[\sum_{i=1}^{N_h} (\mathbf{y}_{hi} - \bar{\mathbf{Y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{Y}}_h)' \otimes (\mathbf{y}_{hi} - \bar{\mathbf{Y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{Y}}_h)' \right] \mathbf{D}_G^{+'}.$$

Observe that the asymptotic means and covariance matrices of the asymptotically normality distributions of $\bar{\mathbf{y}}_h$, $\text{vech } \mathbf{S}_h$, $\text{vec } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$ and $\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$ are in terms of the populations parameters $\bar{\mathbf{Y}}_h$, $\text{vech } \mathbf{S}_h$, \mathfrak{M}_h^4 and \mathbf{M}_h^4 ; then, from Rao (1973, iv), pp. 388-389), approximations of asymptotic distributions can be obtained using consistent estimators instead of population parametrers. In what follows, the following substitutions are used:

$$\bar{\mathbf{Y}}_h \rightarrow \bar{\mathbf{y}}_h, \quad \text{vech } \mathbf{S}_h \rightarrow \text{vech } \mathbf{s}_h, \quad \mathfrak{M}_h^4 \rightarrow \mathbf{m}_h^4 \quad \text{and} \quad \mathbf{M}_h^4 \rightarrow \mathbf{m}_h^4 \quad (14)$$

where

$$\mathbf{m}_h^4 = \frac{1}{n_h} \mathbf{D}_G^+ \left[\sum_{i=1}^{n_h} (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)' \otimes (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)' \right] \mathbf{D}_G^{+'},$$

and

$$\mathbf{m}_h^4 = \frac{1}{n_h} \left[\sum_{i=1}^{n_h} (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)' \otimes (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)' \right].$$

4 OPTIMUM ALLOCATION IN MULTIVARIATE STRATIFIED RANDOM SAMPLING VIA STOCHASTIC MATRIX MATHEMATICAL PROGRAMMING

When the variances are the objective functions, subject to certain cost function, the optimum allocation in multivariate stratified random sampling can be expressed as the following matrix mathematical programming using a deterministic approach

$$\begin{aligned} & \min_{\mathbf{n}} \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \\ & \text{subject to} \\ & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}, \end{aligned} \tag{15}$$

where \mathbb{N} denotes the set of natural numbers. (15) has been studied in detail by Díaz-García and Ulloa (2008).

Observing that $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$ is in terms of $s_{h_{jk}}$, which are random variables, the optimum allocation of (15) via stochastic mathematical programming can be stated as the following stochastic matrix mathematical programming, see Prékopa (1995) and Stancu-Minasian (1984),

$$\begin{aligned} & \min_{\mathbf{n}} \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \\ & \text{subject to} \\ & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & \text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_k \left(\text{E} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right) \\ & n_h \in \mathbb{N}, \end{aligned} \tag{16}$$

where $\text{E} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)$ and $\text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)$ are given by (12) and (13) respectively.

Observe that $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$ is an explicit function of \mathbf{n} , and so it must be denoted as $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \equiv \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n}))$. Also, assume that $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n}))$ is a positive definite matrix for all \mathbf{n} , $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n})) > \mathbf{0}$. Now, let \mathbf{n}_1 and \mathbf{n}_2 be two possible values of the vector \mathbf{n} and, recall that, for \mathbf{A} and \mathbf{B} positive definite matrices, $\mathbf{A} > \mathbf{B} \Leftrightarrow \mathbf{A} - \mathbf{B} > \mathbf{0}$.

Then, proceeding as Díaz-García and Ulloa (2008) the stochastic solution of (16) is reduced to the following stochastic uniobjective mathematical programming problem

$$\begin{aligned} & \min_{\mathbf{n}} f \left(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \\ & \text{subject to} \\ & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & \text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_k \left(\text{E} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right) \\ & n_h \in \mathbb{N}, \end{aligned} \tag{17}$$

where the function f is such that: $f : \mathcal{S} \rightarrow \mathbb{R}$,

$$\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n}_1)) < \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n}_2)) \Leftrightarrow f \left(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n}_1)) \right) < f \left(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n}_2)) \right). \tag{18}$$

with $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n})) \in \mathcal{S} \subset \Re^{G(G+1)/2}$ and \mathcal{S} is the set of positive definite matrices.

Unfortunately or fortunately the function $f(\cdot)$ is not unique. Same alternatives for $f(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}(\mathbf{n})))$ are $\text{tr}(\cdot)$, $|\cdot|$, $\lambda_{\max}(\cdot)$, where λ_{\max} is the maximum eigenvalue, $\lambda_{\min}(\cdot)$, where λ_{\min} is the minimum eigenvalue, $\lambda_j(\cdot)$, where λ_j is the j -th eigenvalue, among others.

Note that (17) is a stochastic uniobjective mathematical programming then, any technique of stochastic uniobjective mathematical programming can be applied, for example:

Point $\mathbf{n} \in \mathbb{N}^H$ is the expected modified value solution to (17) if it is an efficient solution in the **Pareto**¹ sense to following deterministic uniobjective mathematical programming problem

$$\begin{aligned} \min_{\mathbf{n}} \quad & k_1 \text{E} \left(f \left(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right) + k_2 \sqrt{\text{Var} \left(f \left(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right)} \\ \text{subject to} \quad & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}, \end{aligned} \tag{19}$$

Here k_1 and k_2 are non negative constants, and their values show the relative importance of the expectation and the covariance matrix $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$. Some authors suggest that $k_1 + k_2 = 1$, see Rao (1979, p. 599). Observe that if k_1 and k_2 are such that $k_1 = 1$ and $k_2 = 0$ in (19), the resulting method is known as the E-model. Alternatively, if $k_1 = 0$ and $k_2 = 1$, the method is called the V-model, see Charnes and Cooper (1963), Prékopa (1995) and Uryasev and Pardalos (2001).

Alternatively, the point $\mathbf{n} \in \mathbb{N}^H$ is a minimum risk solution of the aspiration level τ to the problem (17) (also termed P-model, see Charnes and Cooper (1963)) if its is an efficient solution in the Pareto sense of the uniobjective stochastic optimization problem

$$\begin{aligned} \min_{\mathbf{n}} \quad & \text{P} \left(f \left(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \leq \tau \right) \\ \text{subject to} \quad & \mathbf{c}'\mathbf{n} + c_0 = C \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, H \\ & n_h \in \mathbb{N}. \end{aligned} \tag{20}$$

In Section 5 the solution is studied for the case when $f = \text{tr} \left(\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)$ and the case when $f = \left| \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right|$. These solutions are implemented in the context of problems (19) and (20).

Finally, note that so far, the cost constraint $\sum_{h=1}^H c_h n_h + c_0 = C$ has been used in every stochastic mathematical programming method. However, in diverse situations, this cost restriction could represent existing restrictions on the availability of man-hours for carrying out a survey, or restrictions on the total available time for performing the survey, etc. These limitations can be established by using the following constraint, see Arthanari and Dodge

¹For the sampling context, observe that in matrix mathematical programming problems, there rarely exists a point \mathbf{n}^* which is considered as a minimum. Alternatively, it say that $f^*(\mathbf{x})$ is a *Pareto point* of $f(\mathbf{n}) = (f_1(\mathbf{n}), \dots, f_G(\mathbf{n}))'$, if there is not other point $f^1(\mathbf{n})$ such that $f^1(\mathbf{n}) \leq f^*(\mathbf{n})$, i.e. for all j , $f_j^1(\mathbf{n}) \leq f_j^*(\mathbf{n})$ and $f^1(\mathbf{n}) \neq f^*(\mathbf{n})$.

Table 1: Variances, covariances and the number of units within each stratum

Stratum	N_h	Variance		Covariance
		BA	Vol.	
1	11 131	1 557	554 830	28 980
2	65 857	3 575	1 430 600	61 591
3	106 936	3 163	1 997 100	72 369
4	72 872	6 095	5 587 900	166 120
5	78 260	10 470	10 603 000	293 960
6	51 401	8 406	15 828 000	357 300
7	24 050	20 115	26 643 000	663 300
8	46 113	9 718	13 603 000	346 810
9	102 985	2 478	1 061 800	39 872

(1981):

$$\sum_{h=1}^H n_h = n.$$

5 APPLICATION

The input information was taken from Arvanitis and Afonja (1971) in which they describe a forest survey conducted in Humboldt County, California. The population was subdivided into nine strata on the basis of the timber volume per unit area, as determined from aerial photographs. The two variables included in this example are the basal area (BA)² in square feet, and the net volume in cubic feet (Vol.), both expressed on a per acre basis. The variances, covariances and the number of units within stratum h are listed in Table 1.

For this example, the matrix optimisation problem under approach (17) is

$$\begin{aligned}
& \min_{\mathbf{n}} f \left(\begin{array}{cc} \widehat{\text{Var}}(\bar{y}_{ST}^1) & \widehat{\text{Cov}}(\bar{y}_{ST}^1, \bar{y}_{ST}^2) \\ \widehat{\text{Cov}}(\bar{y}_{ST}^2, \bar{y}_{ST}^1) & \widehat{\text{Var}}(\bar{y}_{ST}^2) \end{array} \right) \\
& \quad \text{subject to} \\
& \quad \sum_{h=1}^9 n_h = 1000 \\
& \quad 2 \leq n_h \leq N_h, \quad h = 1, \dots, 9 \\
& \quad \text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_3 \left(\text{E} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right), \text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right) \\
& \quad n_h \in \mathbb{N}.
\end{aligned} \tag{21}$$

5.1 Solution when $f(\cdot) \equiv \text{tr}(\cdot)$

Note that by (11), (12) and (13)

$$\text{tr Cov}(\bar{\mathbf{y}}_{ST}) \sim \mathcal{N}(\text{E}(\text{tr Cov}(\bar{\mathbf{y}}_{ST})), \text{Var}(\text{tr Cov}(\bar{\mathbf{y}}_{ST})))$$

²In forestry terminology, ‘Basal area’ is the area of a plant perpendicular to the longitudinal axis of a tree at 4.5 feet above ground.

where

$$\begin{aligned} \mathbb{E} \left(\widehat{\text{tr Cov}}(\bar{\mathbf{y}}_{ST}) \right) &= \sum_{j=1}^G \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \frac{n_h}{n_h - 1} S_{h_j}^2, \\ \text{Var} \left(\widehat{\text{tr Cov}}(\bar{\mathbf{y}}_{ST}) \right) &= \sum_{j=1}^G \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right)^2 \frac{n_h}{(n_h - 1)^2} \left(m_{h_j}^4 - (S_{h_j}^2)^2 \right), \end{aligned}$$

and

$$m_{h_j}^4 = \frac{1}{N_h} \left[\sum_{i=1}^{N_h} \left(y_{hi}^j - \bar{Y}_h^j \right)^4 \right].$$

Therefore, considering the substitutions (14), the equivalent deterministic uniobjective mathematical programming problem to stochastic mathematical programming (21) via the modified E -model is

$$\begin{aligned} \min_{\mathbf{n}} \quad & k_1 \widehat{\mathbb{E}} \left(\widehat{\text{tr Cov}}(\bar{\mathbf{y}}_{ST}) \right) + k_2 \sqrt{\widehat{\text{Var}} \left(\widehat{\text{tr Cov}}(\bar{\mathbf{y}}_{ST}) \right)} \\ \text{subject to} \quad & \sum_{h=1}^9 n_h = 1000 \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ & n_h \in \mathbb{N}, \end{aligned}$$

where

$$\widehat{\mathbb{E}} \left(\widehat{\text{tr Cov}}(\bar{\mathbf{y}}_{ST}) \right) = \sum_{j=1}^2 \sum_{h=1}^9 \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right) \frac{n_h}{n_h - 1} s_{h_j}^2, \quad (22)$$

$$\widehat{\text{Var}} \left(\widehat{\text{tr Cov}}(\bar{\mathbf{y}}_{ST}) \right) = \sum_{j=1}^2 \sum_{h=1}^9 \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right)^2 \frac{n_h}{(n_h - 1)^2} \left(m_{h_j}^4 - (s_{h_j}^2)^2 \right), \quad (23)$$

and

$$m_{h_j}^4 = \frac{1}{n_h} \left[\sum_{i=1}^{n_h} \left(y_{hi}^j - \bar{y}_h^j \right)^4 \right]. \quad (24)$$

Remark 5.1. Observe that the estimators \bar{y}_h^j , $s_{h_j}^2$ and $m_{h_j}^4$ of \bar{Y}_h^j , $S_{h_j}^2$ and $M_{h_j}^4$ are initially obtained as

- i) a consequence of a pilot study (or preliminary sample) or
- ii) using the corresponding values of the estimators of another variable X correlated to the variable Y .

It is important to have this in mind in the the minimisation step, because for example, the n_h 's that appear in expression (24), are the fixed n_h 's values used in the pilot study. Same comment for the expressions of the estimator \bar{y}_h^j and $s_{h_j}^2$. While the n_h 's that appear in expressions (22) and (23) are the decision variables. \square

Similarly, proceeding as in Díaz García *et al.* (2005), and noting that, if Φ denotes the distribution function of the standard Normal distribution, the objective function in (21) with $f(\cdot) \equiv \text{tr}(\cdot)$ can be written as

$$\min_{\mathbf{n}} \quad \Phi \left(\frac{\tau - \widehat{\text{E}} \left(\text{tr } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)}{\sqrt{\widehat{\text{Var}} \left(\text{tr } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)}} \right).$$

In this way, since minimising the monotonically increasing distribution function is equivalent to minimising the value of the associated random variable, the equivalent deterministic problem to the stochastic mathematical programming (21) via the P -model is

$$\begin{aligned} \min_{\mathbf{n}} \quad & \frac{\tau - \widehat{\text{E}} \left(\text{tr } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)}{\sqrt{\widehat{\text{Var}} \left(\text{tr } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)}} \\ \text{subject to} \quad & \\ & \sum_{h=1}^9 n_h = 1000 \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ & n_h \in \mathbb{N}, \end{aligned}$$

Remark 5.2. When $f(\cdot) \equiv |\cdot|$, this approach consider the following alternative stochastic matrix mathematical programming problem

$$\begin{aligned} \min_{\mathbf{n}} \quad & \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \\ \text{subject to} \quad & \\ & \sum_{h=1}^9 n_h = 1000 \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ & \text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_{2 \times 2} \left(\text{vech } \mathbf{0}_{2 \times 2}, \text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right) \\ & n_h \in \mathbb{N}, \end{aligned} \tag{25}$$

where $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) = \text{vech}^{-1} \left[\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) - \text{E} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right]$ and vech^{-1} is the inverse function of function vech .

In this way (20) is

$$\begin{aligned} \min_{\mathbf{n}} \quad & \left| \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right| \\ \text{subject to} \quad & \\ & \sum_{h=1}^9 n_h = 1000 \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ & \text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \xrightarrow{d} \mathcal{N}_{2 \times 2} \left(\text{vech } \mathbf{0}_{2 \times 2}, \text{Cov} \left(\text{vech } \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right) \\ & n_h \in \mathbb{N}, \end{aligned} \tag{26}$$

Thus, taking into account the substitutions (14), the equivalent deterministic uniobjective mathematical programming problem to the stochastic mathematical programming (26) via the modified E -model is

$$\begin{aligned} \min_{\mathbf{n}} \quad & k_1 \widehat{\mathbb{E}} \left(\left| \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right| \right) + k_2 \sqrt{\widehat{\text{Var}} \left(\left| \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right| \right)} \\ \text{subject to} \quad & \\ & \sum_{h=1}^9 n_h = 1000 \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ & n_h \in \mathbb{N}, \end{aligned}$$

where for $G = 2$ and assuming that $\widehat{\text{Cov}} \left(\text{vech} \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right)$ is such that

$$\widehat{\text{Cov}} \left(\text{vech} \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) = \mathbf{B} \otimes \mathbf{B},$$

it is obtained that, see Delannay and Caër (2000),

$$\widehat{\mathbb{E}} \left(\left| \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right| \right) = |\mathbf{N}|^{1/4} \frac{(-1)}{\sqrt{\pi}} (\Gamma[1/2] - \Gamma[3/2]),$$

and $\widehat{\text{Var}} \left(\left| \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right| \right)$ is

$$= |\mathbf{N}|^{1/2} \left[\frac{2}{\sqrt{\pi}} \left(\Gamma[1/2] - \Gamma[3/2] + \frac{\Gamma[5/2]}{2} \right) - \frac{1}{\pi} (\Gamma[1/2] - \Gamma[3/2])^2 \right],$$

where $\Gamma[\cdot]$ denotes the gamma function,

$$\mathbf{N} = \sum_{h=1}^H \left(\frac{W_h^2}{n_h} - \frac{W_h}{N} \right)^2 \frac{n_h}{(n_h - 1)^2} (\mathbf{m}_h^4 - \text{vec } \mathbf{s}_h \text{vec}' \mathbf{s}_h)$$

and

$$\mathbf{m}_h^4 = \frac{1}{n_h} \left[\sum_{i=1}^{n_h} (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)' \otimes (\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)(\mathbf{y}_{hi} - \bar{\mathbf{y}}_h)' \right],$$

see Remark 5.1.

Similarly, considering (25) and that $f(\cdot) \equiv |\cdot|$, (20) is restated as

$$\begin{aligned} \min_{\mathbf{n}} \quad & \mathbb{P} \left(\left| \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right| \leq \tau \right) \\ \text{subject to} \quad & \\ & \sum_{h=1}^9 n_h = 1000 \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ & \text{vech} \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \stackrel{d}{\rightarrow} \mathcal{N}_{2 \times 2} \left(\text{vech } \mathbf{0}_{2 \times 2}, \text{Cov} \left(\text{vech} \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST}) \right) \right) \\ & n_h \in \mathbb{N} \end{aligned}$$

Then, if Ψ denotes the distribution function of the determinant of $\widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$, the equivalent deterministic problem to the stochastic mathematical programming (21) via the P -model is

$$\begin{aligned} \min_{\mathbf{n}} \quad & \tau |\mathbf{N}|^{1/4} \\ \text{subject to} \quad & \\ & \sum_{h=1}^9 n_h = 1000 \\ & 2 \leq n_h \leq N_h, \quad h = 1, 2, \dots, 9 \\ & n_h \in \mathbb{N}, \end{aligned}$$

where the density of $Z = \widehat{\text{Cov}}(\bar{\mathbf{y}}_{ST})$ is, see Delannay and Caër (2000)

$$\frac{dG(z)}{dz} = g_z(z) = \frac{1}{\sqrt{2}} \exp(z) \left[1 - \text{erf}(\sqrt{2z}) \right], \quad z \geq 0,$$

where $\text{erf}(\cdot)$ is the usual error function defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) dt.$$

□

Table 2 shows the optimisation solutions obtained by some of the methods described in Section 4. Specifically, the solution is presented for the case when the value function is defined as the trace function, $f(\cdot) = \text{tr}(\cdot)$ and for the following stochastic solutions: Modified E -model, E -model, V -model and the P -model. Also, the optimum allocation is included for each characteristic, BA and Vol (the first two rows in Table 2). The last two columns show the minimum values of the individual variances for the respective optimum allocations identified by each method. The results were computed using the commercial software Hyper LINGO/PC, release 6.0, see Winston (1995). The default optimisation methods used by LINGO to solve the nonlinear integer optimisation programs are Generalised Reduced Gradient (GRG) and branch-and-bound methods, see Bazaraa *et al.* (2006). Some technical details of the computations are the following: the maximum number of iterations of the methods presented in Table 2 was 2279 (modified E -model) and the mean execution time for all the programs was 4 seconds. Finally, note that the greatest discrepancy found by the different methods among the sizes of the strata occurred under P -model. Beyond doubt, this is a consequence of the election of the corresponding value of τ needed for the P -model approach.

CONCLUSIONS

It is difficult to suggest general rules for the selection of a method in stochastic matrix mathematical programming (16). These conclusions are sustained in several regards, for example: potentiality, there is an infinite number of possible definitions of the value function $f(\cdot)$; furthermore, the value function approach is not the unique way to restate (16); exist many ways to solve (16) from a stochastic point of view. We believe that this responsibility lies with the person skilled in the particular field and in his/her capacity of discern which function or approach that better reflects and meets the objectives of the study.

Table 2: Sample sizes and estimator of variances for the different allocations calculated

Allocation ^a	n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	$\widehat{\text{Var}}(\bar{y}_{ST}^1)$	$\widehat{\text{Var}}(\bar{y}_{ST}^2)$
BA	10	94	144	136	191	113	81	109	122	5.591	5441.105
Vol	7	62	119	136	200	161	98	134	83	5.953	5139.531
$\widehat{\text{tr Cov}}(\bar{\mathbf{y}}_{ST})$											
Modified E-model	8	46	77	119	191	191	158	161	49	7.312	5593.494
E-model ^b	7	63	119	135	200	160	98	134	84	5.937	5139.645
V-model	8	46	77	119	191	191	158	161	49	7.312	5593.494
P-model ^c	632	9	117	29	46	54	52	49	7	29.746	20820.660

^aThe estimated fourth moment $m_{h_j}^4$ were simulated.

^bWhere $k_1 = k_2 = 0.5$.

^cWhere $\tau = 6000$.

In this paper, the problem of optimal allocation in multivariate stratified sampling was considered. In all sample size problems there is always uncertainty regarding the population parameters and in this work, this uncertainty was incorporated via a stochastic matrix mathematical solution.

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REFERENCES

- Arvanitis, L. G., and Afonja, B. (1971). Use of the generalized variance and the gradient projection method in multivariate stratified sampling. *Biometrics*, 27, 119-127.
- Arthanari, T. S., Dodge Y. (1981). *Mathematical Programming in Statistics*. John Wilwy & Sons, New York.
- Bazaraa, M. S., Sherali, H. D., and Shetty, C. M. (2006). *Nonlinear Programming: Theory and Algorithms*, 3rd Edition. Wiley-Interscience.
- Charnes, A., and Cooper, W. W. (1963). Deterministic equivalents for optimizing and satisficing under chance constraints. *Operation Research*, 11, 18-39.
- Cochran, W. G. (1977). *Sampling Techniques*. Wiley, New York.
- Delannay, R. and Caër, G. Le. (2000). Distribution of the determinant of a random real-symmetric matrix from the Gaussian orthogonal ensemble. *Physical Review E*, 62, 1526-1536.
- Díaz-García, J. A., and Garay Tapia, M. M. (2007). Optimum allocation in stratified surveys: Stochastic programming. *Computational Statistics and Data Analysis*, 51, 3016-3026.

- Díaz García, J. A., Ramos-Quiroga, R. and Cabrera-Vicencio, E. (2005). Stochastic programming methods in the response surface methodology. *Computational Statistics and Data Analysis*, 49, 837-848.
- Díaz-García, J. A., and Ulloa, C. L. (2008). Multi-objective optimisation for optimum allocation in multivariate stratified sampling. *Survey Methodology*, 34(2), 215-222.
- Giri, N. C. (1977). *Multivariate statistical inference*. Academic Press, New York.
- Hájek, J. (1961). Some extensions of the Wald-Wolfowitz-Noether theorem. *The Annals of Mathematical Statistics*, 32, 506-523.
- Magnus, J. R., and Neudecker, H. (1988). *Matrix Differential Calculus with Applications in Statistics and Econometrics*. John Wiley & Sons, New York.
- Muirhead, R. J. (1982). *Aspects of multivariate statistical theory*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons, Inc.
- Prékopa, A. (1978). The use of stochastic programming for the solution of the some problems in statistics and probability. Technical Summary report #1834. University of Wisconsin-Madison, Mathematical Research Center.
- Prékopa, A. (1995). *Stochastic Programming*. Kluwer Academic Publishers, Serie Mathematics and its Applications.
- Rao, C. R. (1973). *Linear Statistical Inference and its Applications* (2nd ed.). John Wiley & Sons, New York.
- Rao, S. S. (1979). *Optimization Theory and Applications*. Wiley Eastern Limited.
- Stancu-Minasian, I. M. (1984). *Stochastic Programming*. Reidel P. Co. Dordrecht.
- Sukhatme, P. V., Sukhatme, B. V., Sukhatme, S., and Asok, C. (1984). *Sampling Theory of Surveys with Applications*. Third edition. Ames, Iowa: Iowa State University Press.
- Thompson, M. E. (1997). *Theory of sample surveys*. Chapman & Hall.
- Uryasev, S., Pardalos, P. M. (2001). *Stochastic Optimization*. Kluwer Academic Publishers.
- Winston, W. L. (1995). *Introduction to mathematical programming: Applications and algorithms*. Duxbury Press.